Stability Number and f-Factors in Graphs

Mekkia KOUIDER

November 3, 2010

km@lri.fr

Abstract

Let $f : X \longrightarrow N$ be an integer function. An f-factor is a spanning subgraph of a graph $G = (X, E)$ whose vertices have degrees defined by f . In this paper, we prove a sufficient condition for the existence of a f-factor which involves the stability number, the minimun degree of G or the connectivity of the graph.

Keywords: *Factor, stability number, connectivity, toughness, minimum degree.*

1 Introduction

We consider simple graphs without loops. For notation and graph theory terminology we follow in general [10]. Let G be a graph with vertex set X and edge set $E(G)$. Denote by $d_G(x)$ the degree of a vertex x in G, and by $\delta(G)$ the minimum degree of G. A *spanning subgraph* of G is a subgraph of G with vertex set X. Let $f : X \longrightarrow N$ be an integer function. For any subset A of X, we denote by $f(A)$ the sum $\sum_{x\in A} f(x)$. A spanning subgraph H of a graph G such for every vertex x, $d_H(x) = f(x)$, is called an f-factor of G. Let a, b be fixed integers. A spanning subgraph F of G is called an $[a, b]$-*factor* of G if $a \leq d_F(x) \leq b$ for all $x \in X$.

For $S \subseteq X$, let $|S|$ be the number of vertices in S and let $G[S]$ be the subgraph of G induced by S. We write $G - S$ for $G[X\backslash S]$. A set $S \subseteq X$

is called independent if $G[S]$ has no edges. Denote by $\alpha(G)$ the stability number of a graph G, by $\kappa(G)$ its vertex connectivity. For any vertex $v \in X$, the *open neighborhood* of v is the set $N(v) = \{u \in X \backslash uv \in E(G)\}$; for a set $A \subseteq X$, $N_G(A)$ denotes the set of neighbors in G of vertices in A. Given disjoint subsets $A, B \subseteq X$, we write $e(A, B)$ for the number of edges in G with one extremity in A and the other one in B.
If S is a cutset, let $h'(G - S)$ be the number of components C of $G - S$ such that $\sum_{x \in C} f(x)$ is odd.

Let t be a nonnegative real number. We say that G is t odd-tough if for each cutset S, $h'(G - S) \leq |S|/t$. We remark that if G is t tough then G is t odd-tough.

2 Known Results

Given a graph $G = (X, E)$, an application f and a cuple of disjoint subsets of X, we recall that an *odd component* C of $G - (S \cup T)$ is a component C such that odd.

Many authors have investigated f-factors , see for example [5]. Tutte ([6]) gave the well-known necessary and sufficient condition for existence of an f-factor.
Condition [9] *A graph $G = (X, E)$ has an f-factor if and only if*
1) $\delta(S, T) = f(S) - f(T) + \sum_{v \in T} d_{G \backslash S}(v) - h(S, T) \geq 0$, disjoint subsets S and T of X
where $h(S, T)$ is the number of odd components of $G - (S \cup T)$
2) $\delta(S, T) \equiv f(X)(mod 2)$.

This condition is also a corollary of the (g, f) factor theorem of Lovász in [6]. However, in practise, this condition remains difficult to verify.

Katerinis and Tsikopoulos established a condition on the minimum degree for the existence of f-factors.

Theorem 1 [3] *Let $b \geq a$ two positive integers and let $G = (X, E)$ be a graph with the minimum degree δ. Suppose $\delta \geq \dfrac{b.|X|}{a + b}$, and $|X| >$*

$(a+b)(b+a-3)/a$. *If* f *is a function from* X *to* $\{a, a+1, ..., b\}$ *such that* $f(X)$ *is even, then* G *has an* f*-factor.*

In [2], Katerinis has a condition on the toughness of the graph.

Only few results are known which relate the stability number and factors. Nishimura had sufficient condition for a k factor.

Theorem 2 *Let* $r \geq 1$ *be an odd integer, and* G *be a graph of even order. of connectivity* κ. *If* $\kappa \geq (r+1)^2/2$*,and,* $\alpha(G) \leq \dfrac{4r.\kappa}{(r+1)^2}$*, then* G *has an* r*-factor.*

The following result involving the stability number and the minimum degree of a graph was given by M. Kouider and Zbigniew Lonc [4]:

Theorem 3 [4] *Let* $b \geq a+1$ *and let* G *be a graph with the minimum degree* δ. *If* $\alpha(G) \leq 4b(\delta - a+1))/(a+1)^2$*, for* a *odd and* $\alpha(G) \leq 4b(\delta - a+1)/a(a+2)$*, for* a *even.*

then G *has an* $[a, b]$*-factor.*

Cai has shown that

Theorem 4 c *Let* G *be a connected* $K_{1,n}$*-free graph and let* f *be a nonnegative integer-valued fonction on* $V(G)$ *such that* $1 \leq n-1 \leq a \leq f(x) \leq b$ *for every* $x \in V(G)$.

If $f(V(G))$ *is even ,* $\delta(G) \geq b+n-1$ *and* $\alpha(G) \leq \dfrac{4a.(\delta - b - n + 1)}{(n-1)(b+1)^2}$*, then* G *has an* f *factor.*

Note that Cai conjectured that that the condition on the stability $\alpha(G) \leq \dfrac{4a.(\delta - b)}{(b+1)^2}$ is sufficient in connected graphs. We have the following counterexample.

Suppose b is an odd integer and a an integer strictly less than b.

Let G_0 be a connected graph of minimum degree δ at least $(b+1)^3 + b$. Let $p = \dfrac{4a.(\delta - b)}{(b+1)^2}$. In the graph G_0, we suppose there exists S be a cutset

of $k < b$ vertices, such that $G(S)$ is complete and $C_1, ..., C_p$, the connected components of $G - S$, form a family of complete subgraphs of order $\delta + 1$, mutually independent. Furthermore $G(S \cup C_1)$ is complete, and, for each $i \geq 2$, exactly one edge joins S to C_i. So $\alpha(G_0) = p = \dfrac{4a.(\delta - b)}{(b+1)^2}$.

Let us consider the application f on X such that $f(x) = a$ if $x \in S$, and, $f(x) = b$ otherwise.

If a f factor exists we should have $\alpha = c(G-S) \leq a.k$, so $c(G-S)$ should be at most ab. This is not satified as $\alpha = \dfrac{4a.(\delta - b)}{(b+1)^2} > 4a(b+1)$.

One can see the surveys [8] or [5] for other results.

3 Main Results

We have established a new sufficient condition for a graph to have an f-factor; this condition involves the stability number, the minimum degree of the graph.

Theorem 5 *Let $b \geq 2$ be an integer and let $G = (X, E)$ be a connected graph, of minimum degree δ at least b. Let f be a non-negative integer valued function on X, such that for each $x \in X$, $a \leq f(x) \leq b$ and $f(X)$ is even. If $\alpha(G) \leq \dfrac{4a.(\delta - b)}{(b+1)^2}$, and the odd-toughness of G is at least $1/a$, then*
G contains an f-factor.

Furthermore, we get this corollary.

Corollary 1 *Let $b \geq 2$ be an integer and let $G = (X, E)$ be a graph, of minimum degree δ at least b and connectivity κ. Let f be a non-negative integer valued function on X, such that for each $x \in X$, $a \leq f(x) \leq b$ and $f(x)$ is even. If $\alpha(G) \leq \dfrac{4a.(\delta - b)}{(b+1)^2}$, then*
G contains an f-factor.

Corollary 2 *Let $b \geq 2$ be an integer and let $G = (X, E)$ be a graph, of minimum degree δ at least b and connectivity κ. Let f be a non-negative integer valued function on X, such that for each $x \in X$, $a \leq f(x) \leq b$ and $f(X)$ is even. If $\alpha(G) \leq min(\frac{4a.(\delta - b)}{(b+1)^2}, a\kappa)$, then*

G contains an f-factor $\bullet$

The condition $\alpha(G) < \dfrac{4a.(\delta - b)}{(b+1)^2} + 1$ is necessary if $b > 2a$. Let $\alpha > \delta > b > r$ be four integers. Let us consider a graph G_1 composed by the join of a complete graph $A = K_{\delta - r+1}$ and B, the disjoint union of α complete graphs of order r. Let f be a function such that

$f(x) = a$ if $x \in X(A)$, $f(x) = b$ if $x \in X(B)$. If an f factor exists we get

$$\alpha(G) \leq \frac{a.(\delta - r + 1)}{r.(b + 1 - r)} .$$

For b odd and $r = (b+1)/2$, we get $\alpha(G) < \dfrac{4a.(\delta - b)}{(b+1)^2} + \dfrac{2a}{b+1}$.

4 Proof of Theorem 4

We set first some usefull lemmas.

Lemma 6 *$\delta(S, T)$ is even.*

Proof Let $\mathcal{I}_1$ (respectively $\mathcal{I}_2$) be the set of even (resp. odd) components of $G - (S \cup T)$. By definition,

$$f(\mathcal{I}_1) \equiv e(\mathcal{I}_1, T), \quad (1)$$

$$f(\mathcal{I}_2) \equiv h(S, T) + e(\mathcal{I}_2, , T), \quad (2)$$

so, by (1) and (2),

$$f(X) = f(S) + f(T) + f(\mathcal{I}_1 + f(\mathcal{I}_2) \equiv f(S) - f(T) + e(G - (S \cup T), T) + h(S, T).$$

As $f(X)$ is even, the conclusion follows.

Lemma 7 *T is non-emptyset.*

Proof

If $T = \emptyset$ and $S = \emptyset$, then $\delta(S,T) = -h = 0$ as G is connected and $f(X)$ is even. If $T = \emptyset$ and S is not empty, then $h(S,T)$ is the number of components of $G - S$ such that $f(C)$ is odd.

Either S is not a cutset, then $h(S,T) \leq 1 \leq a|S|$; or S is a cutset, as G is $1/a$-tough, $h \leq a|S|$.

As $a|S| \leq f(S)$, then $\delta(S,T) = f(S) - h(S,T) \geq f(S) - a|S| \geq 0$.

Proposition 1 *If $\alpha(G) \leq \dfrac{4a.(\delta - b)}{(b+1)^2}$, then*

$$|S| > \delta - b$$

Proof

The proof is by contradiction. As $\delta(S,T) < 0$ and $a \leq f(x) \leq b$ for each x, then

$$(\delta - |S|)|T| + a|S| - b|T| - h < 0,$$

so

$$(\delta - |S| - b)|T| < h - a|S|.$$

If $|S| = \delta - b$, we get $|S| < \dfrac{h}{a} < \dfrac{\alpha(G)}{a} < \dfrac{4(\delta - b)}{9}$. This a contradiction.

Now we assume $|S| < \delta - b$, and we get

$$|T| < \frac{h - a|S|}{(\delta - |S| - b)}.$$

If $h < a|S|$, then $|T| = 0$. As $h < \alpha$, then

$$|T| < \frac{4a}{(b+1)^2}.\frac{((\delta - |b|) - (b+1)^2|S|)}{(\delta - |b| - |S|)}$$

We get

$$|T| < \frac{4a}{(b+1)^2}.(1 - \frac{((b+1)^2/4\ \ - 1).|S|}{(\delta - |b| - |S|)})$$

$$T < \frac{4a}{(b+1)^2}.$$

As $b \geq a$, $T < \dfrac{4a}{(b+1)^2} \leq 1$, so $|T| = 0$. This is in contradiction with Lemma 5.

End of the proof of the theorem

Let h_2 be the number of components of $G - (S \cup T)$ not adjacent to T. As $\delta(S,T) < 0$, we have

$$2|E_T| + |T| + a|S| - (b+1)|T| - h_2 \leq 0, \quad (1)$$

As α_T the stability number of T is at least $\dfrac{|T|^2}{2|E_T| + |T|}$, and $\alpha_T \leq \alpha(G) - h_2$, we get, using (1),

$$\alpha(G) - h_2 \geq \frac{|T|^2}{(b+1)|T| - a|S| + h_2}$$

Let us set $|T| = r.|S|$. Then

$$\alpha(G) - h_2 \geq \frac{r^2.|S|^2}{(b+1)r|S| - a|S| + h_2}$$

$$\alpha(G) - h_2 \geq \frac{r^2.|S|}{(b+1)r - a + h_2/|S|}$$

The minimum of the bound as a function of r is for $r = \dfrac{2(a - h_2/|S|)}{b+1}$. It follows that

$$\alpha(G) - h_2 \geq \frac{4a.|S|}{(b+1)^2} - \frac{4h_2}{(b+1)^2}$$

As by hypothesis $\alpha(G) \leq \dfrac{4a(\delta - b)}{(b+1)^2}$, and $|S| \geq (\delta - b)$, we get

$$h_2 \leq \frac{4h_2}{(b+1)^2}.$$

So $h_2 = 0$. As $\delta \geq b$, then $\delta(S,T) \geq 0$. This is a contradiction with the definition of the cuple S, T.

This ends the proof of the theorem 4 •

References

[1] Cai,J. and Liu,G. *Stability Number and f-factor in $K_{1,n}$-free graphs*, Ars Combinatoria **80** (2008).

[2] Katerinis,P., *Toughness and the existence of factors*, Discrete Maths. **80** (1990), 81-92.

[3] Katerinis,P. and Tsikopoulos,N. , *Minimum degree and f-factors in graphs*, New Zealand J.Math. **29** (2000),33-40.

[4] Kouider, M., and Z. Lonc, *Stability Number and $[a, b]$-factors in graphs*, Journal of Graoh Theory **46** (4) (2004), 254-264.

[5] Kouider, M., and P. D. Vestergaard, *Connected factors in graphs, A survey*, Graphs and Combinatorics **21** (2005), 1-26.

[6] Lovász, L., *Subgraphs with prescribed valencies*, J Combin Theory **9** (1970), 391-416.

[7] Nishimura, T., *Independence number, connectivity, and r-factors*, J Graph Theory **13**(1) (1989), 63-69.

[8] Plummer, M. D., *Graph factors and factorization, A survey,* Discrete Mathematics **307** (7-8) (2007), 791-821.

[9] Tutte, W. T., *Graph factors*, Combinatorica **1** (1981), 70-97.

[10] West, D. B., *Introduction to Graph Theory,* Prentice-Hall, Inc. 1996.